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Implicit differential equations, control, singular systems, descriptor systems, numerical methods, analytical methods, system structure, backward differentiation formulas.

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

Results on the numerical and analytic solution of implicit systems of differential equations and their application to circuit and control problems were developed. In particular, the first general algorithm for the linear time varying case was developed along with an analysis of how to apply it to certain control problems. New structure theorems provide insight on the convergence of backward differentiation formulas and guidelines for their use.

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FINAL SCIENTIFIC REPORT

GRANT NO: AFOSR 84-0240

PERIOD: 16 JULY 1984 to 15 JANUARY 1987

PRINCIPAL INVESTIGATOR: Stephen L. Campbell

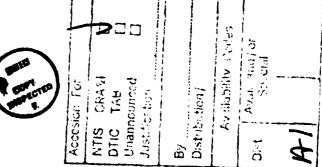Department of Mathematics, Box 8205

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PROJECT DESCRIPTION

Implicit systems of differential equations of the form $F(t,x,x') = 0$ naturally arise in many circuit and control problems, economic models, and the solution of partial differential equations by the method of lines. Implicit systems are also called singular, differential-algebraic, semi-state, constrained, and descriptor. The theory is well understood, and numerical codes exist, for index zero, index one, and linear constant coefficient problems. Higher index implicit systems occur in circuit and control problems. The numerical and analytic behavior of such higher index systems is not well understood and is incomplete. It has recently been shown that traditional methods, such as backward differentiation, need not work on higher index systems. Good characterizations of the solution manifold are often difficult to obtain. This research project is to study the numerical and analytic solution of higher index implicit differential equations. Applications will be made to circuit theory, control theory, and the analysis of numerically ill-conditioned index one systems.

PUBLICATIONS

During the report period the following papers were written as part of this research effort:

P1. S.L. Campbell, *Rank deficient least squares and the numerical solution of linear singular implicit systems of differential equations*, in <u>Linear Algebra and Its Role in Systems Theory</u>, AMS Contempory Mathematics Series, Vol 47(1984), 51-63. (Invited paper)

P2. S.L. Campbell, *Index two linear time varying singular systems of differential equations II*, <u>Circuits, Systems, and Signal Processing</u>, 5(1986), 97-107.

P3. S.L. Campbell, *Nonlinear time-varying generalized state-space systems: an overview*, <u>Proc. 23 IEEE Conference on Decision and Control</u>, (1984), 268-276 (Invited paper)

P4. S.L. Campbell, *Consistent initial conditions for linear time varying singular systems*, in <u>Frequency Domain and State Space Methods for Linear Systems</u>, edited by C.I. Byrnes and A. Lindquist, Elsevier, 1986, 313-318.

P5. K.D. Clark, *Difference methods for the numerical solutions of time-varying singular systems of differential equations*, <u>SIAM J. Algebraic & Discrete Methods</u>, 7(1986), 236-246.

P6. S.L. Campbell, *A general form for solvable linear time varying singular systems of differential equations*, <u>SIAM J. Mathematical Analysis</u>, (to appear)

P7. K.D. Clark, *The numerical solution of some higher index time varying semi-state systems*, <u>Circuits Systems and Signal Processing</u>, (to appear)

P8. K.D. Clark, *The numerical solution of linear time varying singular systems of differential equations by difference methods*, Ph.D. Thesis, North Carolina State University, 1986.

P9. S.L. Campbell, *The numerical solution of higher index linear time varying singular systems of differential equations II*, (preprint) 1986.

P10. S.L. Campbell, *Operator methods and singular control problems* in <u>Operator Methods for Control Problems</u>, Marcel Dekker, 1986, (to appear) (invited paper)

P11. S.L. Campbell. *Bilinear nonlinear descriptor control systems*, <u>Proc. SIAM Conferences on Systems and Signals</u> (to appear) 1986.

P12. K.D. Clark, *A structural form for higher index semistate equations I: theory and applications to circuit and control theory* (preprint) 1986.

Additional references of the principal investigator [Ci] and others [Ri] appear at the end of this report.

PROGRESS ON RESEARCH OBJECTIVES

If a system of differential equations is written in the semi-explicit form as

$$u' = f(u,v,t) \tag{1a}$$

$$0 = g(u,v,t) \tag{1b}$$

then the system (1) is called _index one_ if $\frac{\partial g}{\partial v}$ is nonsingular. In this case the algebraic constraint (1b) may be solved for v, to give $v = \phi(u,t)$ and (1a) becomes an explicit equation $u' = f(u, \phi(u,t), t)$. However, in some control applications [C1], [C7], [R1], [R2], [R7], [R11], [R12] the matrix $\frac{\partial g}{\partial v}$ is always singular. These problems are called _higher index problems_. Definitions of index for the more general system

$$F(x,x',t) = 0 \tag{2}$$

appear in [C5]. Traditionally (1) has been numerically solved using backward differentiation formulas (BDF) [R3], [R4]. Recently, it has been shown that BDF's need not converge for either nonlinear [C5] or linear time varying [R4],

$$A(t)x'(t) + B(t)x(t) = f(t) \tag{3}$$

higher index systems. Prior to the work of the principal investigator, no general method for the numerical solution of (3) existed.

An approach to numerically solve (3) was proposed in [C4]. However, numerous questions on the practical implementation and theoretical implications of this method remained unresolved. The approach was shown in [C4] to be theoretically applicable in several cases of interest, but the range of applicability was uncertain.

The key idea in [C4] was as follows. Suppose that (3) is solvable on an interval I. That is, for every sufficiently smooth input f there is at least one solution x and the solutions for a given f, are uniquely determined by their value at any time $t \in I$. From [C2] it is known that the solution can depend not only on A, B, f but also their derivatives. Thus some differentiation is necessary. Assume that A,B,f are sufficiently smooth on I and at time $\bar{t} \in I$, let $\{A_i\}$, $\{B_i\}$, $\{x_i\}$, $\{f_i\}$ be the Taylor coefficients of A, B, x, and f respectively. These functions do not have to be infinitely differentiable, just sufficiently differentiable. Then for any $j > 0$, any $\bar{t} \in I$, equation (3) implies that the coefficients satisfy

$$
\begin{bmatrix}
A_0 & & \cdot & \cdot & \cdot & \cdot \\
A_1 + B_0 & 2A_0 & \cdot & \cdot & \cdot \\
A_2 + B_1 & 2A_1 + B_0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
A_{j-1} + B_{j-2} & 2A_{j-2} + B_{j-3} & \cdot & \cdot & jA_0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\cdot \\
\cdot \\
x_j
\end{bmatrix}
= -
\begin{bmatrix}
B_0 \\
B_1 \\
\cdot \\
\cdot \\
B_{j-1}
\end{bmatrix}
x_0 +
\begin{bmatrix}
f_0 \\
f_1 \\
\cdot \\
\cdot \\
f_{j-1}
\end{bmatrix}
\quad (4)
$$

or more compactly

$$
\mathcal{E}_j \underline{x}_j = \underline{f}_j . \quad (5)
$$

In [C4] it is shown that for all then known classes of solvable sy, .. that if j
is appropriately chosen, then (5) uniquely determines x_1 even thoug'. .. is not
unique. Since $x_1 = x'(\hat{t})$ it is now possible to define a method to in egrate (3).

This general approach was refined and improved by the principal invastigator
in [P1]. This paper made several contributions. First, it examined the numerical
considerations in solving (5) and showed how to do so in a numerically reliable
manner. Secondly it greatly reduced the computational effort of the method
proposed in [C4] by showing how (5) could be used to define an explicit
differential equation $x' = R(t)x + r(t)$ at each time step. This made possible the
definition of both Adams and Runge-Kutta methods. Due to the smaller sized
coefficient matrices involved, these methods were more computationally efficient
than the higher order Taylor methods of [C4] by an order of magnitude.

One previous difficulty with many of the approaches for solving (3), including
the BDF, has been the lack of general methods for finding the manifold of
consistent initial conditions. In [P4] the principal investigator showed that for
the classes of systems covered by [P1], that the equations (5) could be used to
develop an algorithm for calculating the consistent initial conditions. This is
the first general method for doing so based strictly on the coefficients of A,B, f
and not requiring a series of time-varying coordinate changes.

Although the results reported so far made substantial progress on the analysis
of the linear time varying system (3), several major difficulties remained. Among
these were the lack of either any explicit criteria for solvability or good
descriptions of how to approximate one linear system by another. The full
generality of the technique of [P1] was unknown.

Prior characterizations of solvability, often amounted to being able to solve
the problem. In [P6], the principal investigator gave the first explicit,

verifiable, characterization of solvability in the cases when A,B,f are either real analytic or 3n-times differentiable. Necessary conditions are also given if A,B,f are 2n-times differentiable. These characterizations are stated directly in terms of pointwise rank conditions involving the arrays in (4). Thus they may be verified without performing any coordinate changes.

One consequence of [P6] is a proof that the numerical method of [P1] works for all smooth linear time varying implicit solvable systems (3) and [P4] can always be used to find the initial conditions.

Another consequence of [P6] is a characterization of when the solution manifolds of two linear time varying singular systems are close. This is the first general approximation result to appear for the linear time varying case and opens the door to the development of guidelines for model simplification.

The papers [P1], [P4], [P6] finally put the numerical and analytic solution of (3) on a firm footing. Techniques, for the first time, were now available to study the approximation and simplification of general linear time varying implicit systems.

The method just discussed pays a price for its extra generality. It requires more computational effort at each step than, for example, a BDF method. However, many circuit and control problems have special structure, for example semi-explicit form as in (1). In [P9], the principle investigator showed how to exploit this structure to achieve an additional order of magnitude reduction in computational effort. Also, in [P9] an example was given that showed that the algorithmic changes which lead to the reduction also enabled one to use the method to find a singular arc in a control problem that did not meet all the assumptions for the theory of [P6]. This suggests that the general approach may have even greater applicability than that shown in [P6]. This problem is currently under investigation by a new research assistant.

Many control problems are nonlinear. The results of [P6] provide techniques that can be utilized on some of these problems. In [P11], the principal investigator studied the control problem

$$A(t)x'(t) + (B(t) + u(t)C(t))x(t) = D(t)v(t) \qquad (6)$$

Algorithms were given to determine when the solution x of (6) is a continuous function of the control u and its derivative.

Petzold and Gear [R4], [R5] have shown that most higher index systems (3) can be changed, at least locally, to an index two system by the use of linear time varying coordinate changes. Thus the index two case is of considerable interest. In [C2] the principal investigator had given both sufficient conditions under which an index two linear time varying singular system was solvable and a characterization of the solutions. In [P2] the principal investigator weakened the assumptions of [C2] and obtained a similar result. A series of examples in [P2] showed that additional weakening is probably not possible. These examples shed light on the behavior of systems which are index two but have higher index in a different coordinate system and have proved useful in testing numerical algorithms.

There are two natural types of coordinate changes to make when studying (3); letting x = Q(t)y and multiplication of (3) on the left by P(t). That neither of these effect the analysis of [P1], [P4], [P6], [C4] is one reason that approach is so useful.

It is known that the question of whether BDF's converge is unaltered by time varying P and constant Q but can be altered by time varying Q. The research assistant K. Clark has defined a modified backwards differentiation formula (MBDF) in [P5]. This method has several interesting properties. For linear time

invariant systems and those in the standard canonical form [C6] it is the same as the BDF. However, the convergence properties of the MBDF are unchanged by constant P and time varying Q, but can be altered by time varying P. Thus the MBDF methods provide a nice symmetry to the theory. The methods of [P1], [P4], [C4] are unaffected by time varying P and Q. Additional results on when the BDF and MBDF methods work is given in [P7], [P8], [P12]. This unifies and extends some of the work of [R1].

A somewhat surprising development was the construction of a singular system for which BDF converged but after differentiation of the constraints, BDF did not converge [P8]. Since differentiation of constraints is often used in practice, this has implications for the solution of systems with index greater than two.

From this research has emerged a much better understanding of why BDF methods work on some problems and not others. In [P12] is the beginning of the first truely general theory of convergence of BDF.

The increasing interest in implicit systems within the mathematical and engineering communities is illustrated by the recent special sessions held on this topic and invitations to the principal investigator. Several of these are described in the travel section. In addition, the paper [P2] will appear in a special issue of Circuits Systems & Signal Processing on semistate equations.

The research of the principal investigator and others on singular systems is beginning to have the desired effect of making this approach accessible and useful to scientists and engineers. Examples are the recent papers on robotics [R7], chemical kinetics [R8], moving grids in partial differential equations [R8], [R10], flight control [R1], [R2], mechanics [R11], [R12], and other applications involving the method of lines [R8], [R10], [R12].

The research reported here has also laid the groundwork for consideration of new approaches to attacking important nonlinear problems. These approaches are now being examined.

TRAVEL

Travel funds from this grant were used to enable the principal investigator to attend the Society for Industrial and Applied Mathematics (SIAM) 1984 National Summer Meeting in Seattle, Washington, July 16-20. A paper concerning this research was presented. The meeting provided an opportunity to discuss the proposed research with experts from numerical analysis, control theory, and differential equations.

July 30 - August 3, 1984; the principal investigator attended and gave an invited paper [P1] at the American Mathematical Society (AMS) - SIAM Research Conference on "Linear Algebra and Its Role in Systems Theory". Researchers from Engineering and Mathematics with an interest in control theory, systems theory, linear algebra, and numerical linear algebra were present. Recent results by the principal investigator on the linear algebra problems that arise in the numerical solution of the implicit systems (3) were presented.

December 12-14, 1984; the principal investigator attended and gave an invited paper [P3] at the 23rd IEEE Conference on Decision and Control as part of a special session on generalized state space systems.

June 9-15, 1985; the principal investigator attended and gave an invited talk [P4] at the 1985 Mathematical Theory of Networks and Systems Conference in Stockholm, Sweden.

June 18-23, 1985; the principal investigator visited the University of Bologna, Italy. He gave a seminar on recent results on singular systems and had mathematical discussions with Professor Angelo Favini about operator theoretic questions involving implicit systems of differential equations. While considering

these questions, the principal investigator was able to prove, by operator theoretic means, one of the key lemmas in [P6]. This trip was partly supported by AFOSR and partly by the Italian government.

June 24-26, 1985; the principal investigator attended the 1985 SIAM National meeting in Pittsburgh, PA. He gave an invited talk in the special session on differential-algebraic equations.

January 7-11, 1986; the principal investigator attended the 1986 National American Mathematical Society Meeting in New Orleans. He gave an invited talk in [P10] in the special session on Operator Methods in Control.

July 27-August 2, 1986; the principal investigator attended and presented a paper at the 1986 Ordinary Differential Equations Conference in Albuquerque. This was a large meeting devoted to the numerical solution of differential equations. There were several people attending from universities, national laboratories, and corporations that were interested in singular systems.

August 12-14, 1986; the principal investiagor attended and presented a paper at the SIAM Meeting on Linear Algebra in Signals, Systems, and Control in Boston, Mass. He was also on the organizing committee for this meeting. Approximately 400 mathematicians and engineers attended.

In addition to AFOSR sponsored travel, the principal investigator made two trips to Lawrence Livermore National Laboratory (LLNL), Livermore, California where he is a Participating Guest with a DOE Q clearance. The trips were July 19-20, 1986, and December 11-13, 1986. The principal investigator consulted with Dr. Linda Petzold of the Mathematical Computation Group, Dr. K. Brenan who was visiting from Aerospace Corporation, and B. Leimkuhler from the University of Illinois. The discussions were on the numerical solution of differential algebraic systems that arise in various applications. These trips were paid for by LLNL.

OTHER PROJECT PERSONNEL

Research Assistant:

The Research Assistant, Mr. Kenneth Clark, worked on a Ph.D. in Applied
Mathematics under the direction of the principal investigator. Mr. Clark developed
some nice results on the relationship between the convergence of backward
differentiation formulas and the types of transformations needed to put an implicit
system into a canonical form. These results are reported in [P5], [P7], [P8],
[P12]. He graduated in June, 1986. An abstract of his thesis follows.

Abstract of Research Ph.D. Thesis

TITLE: Difference Methods for the Numerical Solution of Nonautonomous Singular
Linear Systems.

ABSTRACT. Several aspects of the numerical and analytical treatment of singular
linear systems

$$A(t)x' + B(t)x = f(t)$$

are considered. Here $A(t)$ is assumed to be identically singular, and the
differential equation is considered on the closed interval $[0,T]$. First, a class
of modified difference schemes is derived for the numerical solution of the system.
These methods are analyzed on several classes of problems which frequently arise,
and in particular their convergence and stability is proved for the classes
considered. The classes we examine include those transformable to constant
coefficients, implicitly defined index one systems, and index two systems in
semi-explicit form. We show that all of the classes investigated in this paper can
be treated in a more general framework involving a special case of the standard

canonical form for time varying singular linear systems, and that the structure of the form is the key element in the analysis.

ADDITIONAL REFERENCES

Of the Principal Investigator

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C2. S. L. Campbell, *Index two linear time-varying singular systems of differential equations*, SIAM J. Algebraic and Discrete Methods, 4(1983), 237-243.

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Others

R1. K. E. Brenan, *Stability and convergence of difference approximations for higher index differential algebraic systems with applications in trajectory control*, Ph.D. Thesis, Univ. of Calif., Los Angeles 1983.

R2. K. E. Brenan, *Numerical simulation of trajectory prescribed path control problems*, IEEE Trans. Aut. Control, AC-31, (1983), 266-269.

R3. C. W. Gear, *Simultaneous numerical solution of differential-algebraic equations*, IEEE Trans. Circuit Theory, CT-18, (1971), 89-95.

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R6.   C. W. Gear, G. K. Gupta, and B. Leimkuhler, *Automatic integration of Euler-Lagrange equations with constraints*, J. Comp. Appl. Math, 12 & 13, 1985, 77-90.

R7.   N. Harris McClamroch, *Singular systems of differential equations as dynamic models for constrained robot systems*, Tech. Report RSD-TR-2-86, Center for Research on Integrated Manufacturing, College of Engineering, Univ. of Mich., 1986.

R8.   R. J. Kee, L. R. Petzold, M. D. Sincoke, and J. F. Grear, *Implicit methods in combustion and chemical kinetics modeling*, in Multiple Time Scales, ed. J. Brackbill, B. Cohen, Academic Press, Inc., 1985, 113-144.

R9.   L. R. Petzold, *Differential/algebraic equations are not ODE's*, SIAM J. Scientific and Stat. Comp., 3(1982), 367-384.

R10.   L. R. Petzold, *Observations on an adaptive moving grid method for one-dimensional systems of partial differential equations*, Lawrence Livermore National Laboratory Report UCRL-94629, 1986.

R11.   P. Lotstedt and L. R. Petzold, *Numerical solution of nonlinear differential equations with algebraic constraints, I: Convergence results for the backward differentiation formulas*, Math. Comp. 46(1986) 491-516.

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